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# CONSERVATION LAWS AND GROUP PROPERTIES OF EQUATIONS OF ISENTROPIC GAS MOTION

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*A system of equations of isentropic gas motion with $n \geqslant 2$ is classified in terms of zero-order conservation laws with the use of the method of $\boldsymbol{A}$-operators. New conservation laws are found to be valid only for potential isentropic motion of the Chaplygin gas. In this case, the greatest number of nontrivial conservation laws is obtained, with $n$ scalar conservation laws being nonlocal. Additional properties of symmetry of the considered equations associated with these conservation laws are indicated.*

**Key words:** *conservation law, classification of equations of isentropic gas motion, nonlocal conservation laws, nonlocal symmetries, Chaplygin gas.*

**Introduction.** The search for conservation laws for systems of differential equations is discussed in many publications. In particular, Shmyglevskii [1] obtained a full system of zero-order conservation laws for equations of motion of a perfect gas in a three-dimensional formulation. Ibragimov [2] found additional conservation laws for equations of motion of a polytropic gas by applying point symmetry operators to the classical conservation laws. Ibragimov [2] also derived an additional conservation law for a system of equations of potential isentropic motion of a polytropic gas.

In the present work, a system of equations of isentropic gas motion with $n \geqslant 2$ is classified in terms of zero-order conservation laws with the use of the method of $\boldsymbol{A}$-operators [3]. It is demonstrated that the condition of an isentropic character of the flow does not generate new conservation laws in this system (in contrast to the conventional system of equations of gas motion). A system of vortex-free isentropic gas motion and a system of equations of potential isentropic gas motion are classified in terms of the zero-order conservation laws by the method of $\boldsymbol{A}$-operators. For the latter system of equations, the greatest number of nontrivial conservation laws is observed in the case of the Chaplygin gas, with $n$ scalar conservation laws being nonlocal. A group classification is performed for the system of equations of potential isentropic gas motion, which allows the set of nontrivial zero-order conservation laws to be extended. It is found that the system for the Chaplygin gas admits the greatest Lie group of transformations.

Isentropic gas motion is described by the equations

$$\boldsymbol{u}_t + (\boldsymbol{u} \cdot \nabla) u + f(\rho) \nabla \rho = \boldsymbol{0}, \qquad \rho_t + \boldsymbol{u} \cdot \nabla \rho + \rho \operatorname{div} \boldsymbol{u} = 0, \tag{1}$$

where $t$ is the time, $\boldsymbol{u} = \boldsymbol{u}(t, \boldsymbol{x})$ is the velocity vector, $\rho = \rho(t, \boldsymbol{x})$ is the density, $\boldsymbol{x} = (x^1, \ldots, x^n) \in \mathbb{R}^n$ $(n \geqslant 2)$, $f = f(\rho) = c^2/\rho$, and $c = c(\rho) > 0$ is the velocity of sound.

In the case of vortex-free isentropic motion, system (1) is supplemented by the equation

$$\nabla \Lambda \boldsymbol{u} = \boldsymbol{0}. \tag{2}$$

Introducing the potential $\varphi = \varphi(t, \boldsymbol{x})$ of the velocity vector

$$\boldsymbol{u} = \nabla \varphi \tag{3}$$

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and integrating the momentum equation in system (1), we obtain the Cauchy–Lagrange integral

$$\varphi_t + |\nabla\varphi|^2/2 + i(\rho) = 0, \tag{4}$$

where $i = i(\rho)$ is the specific enthalpy [$i'(\rho) > 0$].

Using the method of $\boldsymbol{A}$-operators, we solve the problem of classification of Eqs. (1)–(4) in terms of zero-order conservation laws and establish additional properties of symmetry for these equations, which are caused by new conservation laws.

**1. Conservation Laws.** The zero-order conservation law for a system of $(S)$ first-order differential equations with independent variables $t$ and $\boldsymbol{x}$ and dependent variables $\boldsymbol{u}$, $\rho$, and $\varphi$ is a relation of the form [2]

$$(\boldsymbol{D}\cdot\boldsymbol{A})_{(S)} = 0,$$

where $\boldsymbol{A} = \boldsymbol{A}(t, \boldsymbol{x}, \boldsymbol{u}, \rho, \varphi) = (A^0, A^1, \ldots, A^n)$, $\boldsymbol{D} = (D_0, D_1, \ldots, D_n)$, $D_i = D_{x^i}$ is the operator of total differentiation with respect to the variable $x^i$ $(i = 0, 1, 2, \ldots, n)$, and $\boldsymbol{x}^0 = t$. In gas dynamics, the physical meaning of the conservation law $\boldsymbol{A} = (A^0, A^1, \ldots, A^n)$ is determined by the component $A^0$, which is the density in the conservation law, and by the flow vector $\boldsymbol{B} = \boldsymbol{A}_1 - A^0\boldsymbol{u}$, where $\boldsymbol{A}_1 = (A^1, \ldots, A^n)$.

As the generating conservation law $\boldsymbol{A}$ [3], we use the momentum conservation law

$$A^0 = \rho\boldsymbol{u}\cdot\boldsymbol{a}, \qquad \boldsymbol{B} = p\boldsymbol{a},$$

where $\boldsymbol{a}$ is a constant fixed unit vector and $p$ is the pressure.

The results of the classification performed allow us to draw the following conclusions.

1. The set of nontrivial zero-order conservation laws for system (1), which describes isentropic gas motion, is restricted in the case of an arbitrary function $f(\rho)$ to the classical conservation laws. This set is extended only for a polytropic gas with an adiabatic index equal to $(n + 2)/n$. In this case, the conservation laws derived in [2] are also valid:

$$A^0 = t(\rho|\boldsymbol{u}|^2 + np) - \rho\boldsymbol{x}\cdot\boldsymbol{u}, \quad \boldsymbol{B} = p(2t\boldsymbol{u} - \boldsymbol{x}), \quad A^0 = t^2(\rho|\boldsymbol{u}|^2 + np) - \rho\boldsymbol{x}\cdot(2t\boldsymbol{u} - \boldsymbol{x}), \quad \boldsymbol{B} = 2tp(t\boldsymbol{u} - \boldsymbol{x}). \tag{5}$$

2. The set of nontrivial zero-order conservation laws for system (1), (2), which describes vortex-free isentropic gas motion, is restricted in the case of an arbitrary function $f(\rho)$ to the classical conservation laws and the conservation law

$$A^0 = \boldsymbol{u}\cdot\operatorname{div}\Omega(t, \boldsymbol{x}), \qquad B = (|\boldsymbol{u}|^2/2 + i(\rho))\operatorname{div}\Omega(t, \boldsymbol{x}) + \Omega_t(t, \boldsymbol{x})\langle\boldsymbol{u}\rangle - (\boldsymbol{u}\cdot\operatorname{div}\Omega(t, \boldsymbol{x}))\boldsymbol{u},$$

where $\Omega(t, \boldsymbol{x})$ is an arbitrary antisymmetric tensor of rank 2 in $\mathbb{R}^n$, which is a consequence of the vortex-free character of motion. This set is extended only for a polytropic gas with an adiabatic index equal to $(n + 2)/n$. In this case, the conservation laws (5) are also valid.

3. The set of nontrivial zero-order conservation laws for system (1)–(4), which describes potential isentropic gas motion, is restricted in the case of an arbitrary function $f(\rho)$ to the classical conservation laws. Additional conservation laws appear only at $f(\rho) = \alpha\rho^{\gamma-2}$ ($\alpha \neq 0$ and $\gamma$ are arbitrary constants).

For $\gamma \neq -1$ and $(n + 2)/n$, the conservation law

$$\begin{gathered} A^0 = \rho[(\gamma(n+2) - n)t|\boldsymbol{u}|^2/2 - (\gamma n - (n+2))(ti(\rho) + \varphi) + (\gamma+1)(ntp/\rho - (\boldsymbol{x}\cdot\boldsymbol{u}))], \\ \boldsymbol{B} = 2(n+1)t\rho^2 i'(\rho)\boldsymbol{u} - (\gamma+1)p(nt\boldsymbol{u} - \boldsymbol{x}) \end{gathered} \tag{6}$$

is also valid, which is equivalent to the conservation law given in [2] for a polytropic gas with the equation of state $p = \alpha\rho^\gamma$ ($\alpha = \text{const} \neq 0$) at $\gamma \neq \pm 1$.

For $\gamma = (n + 2)/n$, the conservation laws (5) are also valid.

For $\gamma = -1$ (Chaplygin gas), in addition to the conservation law (6), the following conservation law is also valid:

$$A^0 = \rho\boldsymbol{b}\cdot[(|\boldsymbol{u}|^2/2 - p/\rho - (\rho i(\rho))')\boldsymbol{x} - \varphi\boldsymbol{u}], \qquad \boldsymbol{B} = p(\boldsymbol{x}\cdot\boldsymbol{b})\boldsymbol{u} + \varphi\rho^2 i'(\rho)\boldsymbol{b} \tag{7}$$

($\boldsymbol{b}$ is an arbitrary constant unit vector).

Formulas (7) define $n$ independent conservation laws for system (1)–(4). In the case of the Chaplygin gas, therefore, this system has the greatest number of nontrivial zero-order conservation laws.

The conservation law (6) with $\gamma \neq (n+2)/n$ and the conservation law (7) with $\gamma = -1$ are nonlocal conservation laws for system (1) with a nonlocal variable $\varphi$. Thus, system (1) has nontrivial nonlocal zero-order conservation laws depending on the nonlocal variable $\varphi = \varphi(t, \boldsymbol{x})$ if and only if the equation of state of the gas is determined by the relation $f(\rho) = \alpha\rho^{\gamma-2}$, where $\alpha \neq 0$ and $\gamma \neq (n+2)/n$ are arbitrary constants. The corresponding nonlocal conservation laws are determined by formulas (6) and (7). The greatest number of such conservation laws is observed in the case of the Chaplygin gas.

**2. Group Properties.** To answer the question which properties of symmetry of equations that describe potential isentropic gas motion are responsible for the emergence of additional conservation laws, we consider the problem of the group classification of system (1)–(4) with respect to an arbitrary element $f(\rho)$.

The equivalence transformations that retain the differential structure of system (1)–(4) are described by the relations

$$t' = at, \quad \boldsymbol{x}' = \boldsymbol{x}, \quad \varphi' = \frac{\varphi}{a} + abt, \quad \rho' = k\rho, \quad \boldsymbol{u}' = \frac{\boldsymbol{u}}{a},$$

where $a$, $b$, and $k$ ($ak \neq 0$) are arbitrary constants. The arbitrary element is transformed by the formula

$$f'(\rho') = \frac{1}{a^2k}\, f\Big(\frac{\rho'}{k}\Big).$$

The kernel of the basic Lie groups of transformations of system (1)–(4) is generated by the operators

$$\partial_t, \quad \partial_{\boldsymbol{x}}, \quad t\,\partial_t + \boldsymbol{x}\cdot\partial_{\boldsymbol{x}} + \varphi\,\partial_\varphi, \quad t\,\partial_{\boldsymbol{x}} + \partial_{\boldsymbol{u}} + \boldsymbol{x}\,\partial_\varphi, \quad Q\langle\boldsymbol{x}\rangle\cdot\partial_{\boldsymbol{x}} + Q\langle\boldsymbol{u}\rangle\cdot\partial_{\boldsymbol{u}}, \quad \partial_\varphi$$

($Q$ is an arbitrary antisymmetric tensor of rank 2 in $\mathbb{R}^n$) and is (with accuracy to the operator $\partial_\varphi$) a presentation of the Galileo group in the space $\mathbb{R}^{2n+2}(t, \boldsymbol{x}, \boldsymbol{u}, \varphi)$.

The basic group is extended only at $f(\rho) = \alpha\rho^{\gamma-2}$, where $\alpha \neq 0$ and $\gamma$ are arbitrary constants.

For $\gamma \neq \pm 1$ and $(n+2)/n$, system (1)–(4) admits the additional operator

$$t\,\partial_t - \boldsymbol{u}\cdot\partial_{\boldsymbol{u}} - \varphi\,\partial_\varphi - \frac{2}{\gamma-1}\,\rho\,\partial_\rho. \tag{8}$$

For $\gamma = 1$, system (1)–(4) admits the additional operator

$$t\,\partial_\varphi - \rho\,\partial_\rho.$$

For $\gamma = (n+2)/n$, system (1)–(4) admits two additional operators: operator (8) and the operator

$$t^2\,\partial_t + t\boldsymbol{x}\cdot\partial_{\boldsymbol{x}} + (\boldsymbol{x} - t\boldsymbol{u})\cdot\partial_{\boldsymbol{u}} + (1/2)|\boldsymbol{x}|^2\,\partial_\varphi - nt\rho\,\partial_\rho.$$

For $\gamma = -1$ (Chaplygin gas), system (1)–(4) admits the following additional operators: operator (8) and the nonlocal vector operator

$$\boldsymbol{x}\partial_t + \varphi\partial_{\boldsymbol{x}} + (|\boldsymbol{u}|^2/2 - 1/\rho^2)\,\partial_{\boldsymbol{u}} - \boldsymbol{u}(\boldsymbol{u}\cdot\partial_{\boldsymbol{u}} - \rho\,\partial_\rho), \tag{9}$$

where $\varphi = \varphi(t, \boldsymbol{x})$ is the nonlocal variable. Therefore, it is in the case of the Chaplygin gas that system (1)–(4) admits the greatest Lie group of transformations.

The conservation law (5) is (with accuracy to the trivial conservation law) a result of the action of the canonical Lie–Backlund operator [2] equivalent to operator (9) on the momentum conservation law.

**Remark 1.** In the case of the group classification of the system

$$\varphi_t + |\nabla\varphi|^2/2 + i(\rho) = 0, \qquad \rho_t + \nabla\varphi\cdot\nabla\rho + \rho\,\Delta\varphi = 0,$$

which is a subsystem of system (1)–(4), the Chaplygin gas does not possess specific properties.

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## REFERENCES

1. Yu. D. Shmyglevskii, *Analytical Study of Fluid Dynamics* [in Russian], URSS, Moscow (1999).
2. N. Kh. Ibragimov, *Groups of Transforms in Mathematical Physics* [in Russian], Nauka, Moscow (1983).
3. Yu. A. Chirkunov, “Method of ***A***-operators and conservation laws for the4 equations of gas dynamics,” *J. Appl. Mech. Tech. Phys.*, **50**, No. 2, 213–219 (2009).